8 Waves in Uniform Magnetized Media

8.1 Susceptibilities

The first order current can be written

$$\vec{j} = \sum_s \vec{j}_s = \sum_s q_s \int d^3\vec{p}\,\vec{v}_s f_{s1}(\vec{r},\vec{p},t) = -i\omega\epsilon_0 \sum_s \overleftrightarrow{\chi}_s \cdot \vec{E}$$

For Maxwellian distribution

$$f_0(v_\perp, v_\parallel) = \frac{1}{\sqrt{\pi}v_{th\parallel}} \exp\left[-\frac{(v_\parallel - V)^2}{v_{th\parallel}^2}\right] \frac{1}{\pi v_{th\perp}^2} \exp\left(-\frac{v_\perp^2}{v_{th\perp}^2}\right)$$

$$\overleftrightarrow{\chi}_s = \left[\vec{\hat{e}}_\parallel \vec{\hat{e}}_\parallel \frac{2\omega_p^2 V}{\omega k_\parallel v_{th\perp}^2} + \frac{\omega_p^2}{\omega} \sum_{-\infty}^{\infty} e^{-\lambda} \overleftrightarrow{Y}_n(\lambda)\right]_s$$

$$\overleftrightarrow{Y}_n(\lambda) = \begin{pmatrix} \frac{n^2 I_n}{\lambda} A_n & -in(I_n - I_n')A_n & \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n \\ in(I_n - I_n')A_n & \left(\frac{n^2}{\lambda} I_n + 2\lambda I_n - 2\lambda I_n'\right) A_n & ik_\perp \Omega (I_n - I_n') B_n \\ \frac{k_\perp}{\Omega}\frac{nI_n}{\lambda} B_n & -ik_\perp \Omega (I_n - I_n') B_n & \frac{2(\omega - n\Omega)}{k_\parallel v_{th\perp}^2} B_n \end{pmatrix}$$

$$I_n = I_n(\lambda), \quad \lambda = \frac{k_\perp^2 v_{th\perp}^2}{2\Omega} = \frac{1}{2} k_\perp^2 < \rho_L^2 >$$

$$A_n = \frac{1}{\omega}\frac{T_\perp - T_\parallel}{T_\perp} + \frac{1}{k_\parallel v_{th\parallel}} \frac{(\omega - k_\parallel V - n\Omega)T_\perp + n\Omega T_\parallel}{\omega T_\parallel} Z_0(\zeta_n)$$

$$\begin{aligned} B_n &= \frac{1}{k_\parallel}\frac{(\omega - n\Omega)T_\perp - (k_\parallel V - n\Omega)T_\parallel}{\omega T_\parallel} \\ &\quad + \frac{1}{k_\parallel}\frac{\omega - n\Omega}{k_\parallel v_{th\parallel}} \frac{(\omega - k_\parallel V - n\Omega)T_\perp + n\Omega T_\parallel}{\omega T_\parallel} Z_0(\zeta_n) \end{aligned}$$

$$\zeta_n = \frac{\omega - k_\parallel V - n\Omega}{k_\parallel v_{th\parallel}}.$$

For $V = 0$, $T_\perp = T_\parallel$

$$A_n = \frac{1}{k_\parallel v_{th}} Z_0(\zeta_n)$$

$$B_n = \frac{1}{k_\parallel}\left[1 + \zeta_n Z_0(\zeta_n)\right]$$

8.2 Parallel Propagation

For parallel propagation $\vec{k} \parallel \vec{B}_0$, and in the limit $k_\perp \to 0$ ($\lambda \to 0$),

$$e^{-\lambda} \overleftrightarrow{Y}_0(\lambda) = \vec{e}_\parallel \vec{e}_\parallel \frac{2\omega V}{k_\parallel v_{th\perp}^2} B_0$$

$$e^{-\lambda} \overleftrightarrow{Y}_{\pm 1}(\lambda) = \frac{1}{2} \begin{pmatrix} 1 & \pm i & 0 \\ \mp i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} A_{\pm 1}$$

The dispersion relation can be derived as

$$n_\parallel^2 = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \left[\frac{T_\perp - T_\parallel}{T_\parallel} + \frac{(\omega - k_\parallel V \pm \Omega) T_\perp \mp \Omega T_\parallel}{k_\parallel v_{th\parallel} T_\parallel} Z_0 \left(\frac{\omega - k_\parallel V \pm \Omega}{k_\parallel v_{th\parallel}} \right) \right]_s$$

which correspond to cold plasma dispersion relations $n_\parallel^2 = R$ and $n_\parallel^2 = L$, and

$$0 = 1 + \sum_s \frac{2\omega_{ps}^2}{k_\parallel^2 v_{th\parallel}^2} \left[1 + \frac{\omega - k_\parallel V}{k_\parallel v_{th\parallel}} Z_0 \left(\frac{\omega - k_\parallel V}{k_\parallel v_{th\parallel}} \right) \right]_s$$

which corresponds to cold plasma dispersion relation $0 = P$.

For the case $V = 0$, $T_\perp = T_\parallel$, $0 < \Re\omega \ll |\Omega_e|$

$$\frac{k_\parallel^2 c^2}{\omega^2} \simeq 1 - \omega_{pi}^2 \left\{ \frac{1}{\Omega(\omega - \Omega)} + \frac{k_\parallel^2 T}{m\omega(\omega - \Omega)^3} - \frac{i\sqrt{\pi}}{\omega |k_\parallel| v_{th}} \exp\left[- \left(\frac{\omega - \Omega}{k_\parallel v_{th}} \right)^2 \right] \right\}_i .$$

The damping rate (assuming k is real and $|\omega_i| \ll \omega_r$) is given by

$$\frac{\omega_i}{\omega_r} \simeq - \left\{ \frac{\omega^3}{2\Omega^3 - \omega\Omega^2} \frac{\omega_p^4}{k_\parallel^4 c^4} \frac{\sqrt{\pi}\Omega}{|k_\parallel| \omega} \exp\left[- \left(\frac{\omega - \Omega}{k_\parallel v_{th}} \right)^2 \right] \right\}_i .$$

The damping length (assuming ω is real and $|k_i| \ll |k_r|$) is given by

$$\left(\frac{k_i}{k} \right)_\parallel \simeq \left\{ \frac{\omega}{2\Omega} \frac{\omega_p^2}{k_\parallel^2 c^2} \frac{\sqrt{\pi}\Omega}{|k_\parallel| v_{th}} \exp\left[- \left(\frac{\omega - \Omega}{k_\parallel v_{th}} \right)^2 \right] \right\}_i .$$

For $\omega \simeq \Omega_i$

$$n_\parallel^2 = 1 - \frac{\omega_{pi}^2}{\omega \Omega_i} + \left[\frac{\omega_p^2}{\omega k_\parallel v_{th}} Z_0(\zeta_1) \right]_i .$$

Note that $|Z_0(\zeta)|_{\max} = \sqrt{\pi}$ at $\zeta = 0$. For ion cyclotron wave

$$|k_\parallel|^3 < \frac{\sqrt{\pi}\Omega_i \omega_{pi}^2}{c^2 v_{thi}}$$

and $k_\parallel$ does not become infinite as ω approaches Ω_i.

8.3 Cyclotron Harmonic Damping

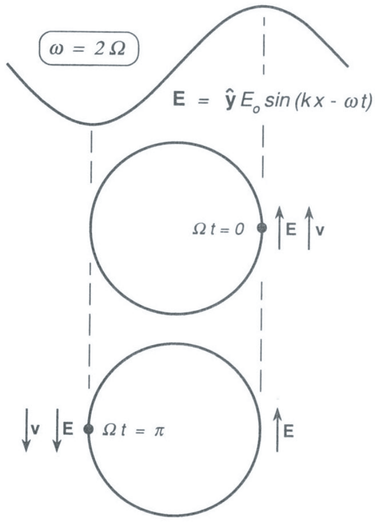
Wave absorption at harmonics of cyclotron frequency ($\omega = n\Omega$) can occur when finite $k_\perp$ effect is taken into account. Consider the case of second harmonic $n = 2$ with $\vec{E} = \vec{y}E\sin(k_x x - \omega t)$, $\omega = 2\Omega$, $k_x = \pi/(2\rho_L)$. The rate of perpendicular kinetic energy increase is given by

$$\frac{dW_\perp}{dt} = \frac{d}{dt}\frac{mv_\perp^2}{2} = m\vec{v}_\perp \cdot \frac{d\vec{v}_\perp}{dt} = q\vec{v}_\perp \cdot \left[\vec{E}_\perp + \vec{v} \times \vec{B}\right] = q\vec{v}_\perp \cdot \vec{E}.$$

Substituting the unperturbed orbit $x = \rho_L \sin(\Omega t + \phi)$ in the expression for $\vec{E}$ gives

$$\begin{aligned}\frac{dW_\perp}{dt} &= qE\Omega\rho_L \cos\left[k_\perp\rho_L \sin(\Omega t + \phi) - \omega t\right]\cos(\Omega t + \phi) \\ &= qE\Omega\rho_L \Re\left[\sum_{n=-\infty}^{\infty} J_n(k_\perp\rho_L)e^{in(\Omega t+\phi)-i\omega t}\right]\Re\left[e^{i(\Omega t+\phi)}\right].\end{aligned}$$

It can be seen that when $\omega = (n \pm 1)\Omega$, $W_\perp$ will keep increasing (or decreasing).



Fg. 1. Schematic showing the mechanism of 2nd harmonic cyclotron damping.

As an example, consider the case of the fast wave at $\omega = 2\Omega_i$. Assuming $|\omega_i| \ll |\omega_r|$, $V_i = 0$, and $\omega \simeq 2\Omega_i$, in which case

$$\Im A_2 \simeq \left\{\frac{\sqrt{\pi}}{|k_\parallel|v_{th\parallel}} \exp\left[-\left(\frac{\omega - 2\Omega_i}{k_\parallel v_{th\parallel}}\right)^2\right]\right\}_i.$$

The wave equation can be written as

$$\begin{pmatrix} L_h + R - 2n_\parallel^2 & iL_h - iR \\ -iL_h + iR & L_h + R - 2n_\parallel^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

where $L_h = L + \Delta L$ and $\Delta L = i(\omega_{pi}^2/\omega)\lambda \Im A_2$. The dispersion relation is given by

$$n_\perp^2 - \frac{(R - n_\parallel^2)(L - n_\parallel^2)}{S - n_\parallel^2} - \frac{1}{2}\left(\frac{R - n_\parallel^2}{S - n_\parallel^2}\right)^2 \Delta L = 0.$$

This can be solved to obtain the damping rate

$$\omega_i \simeq \left(\frac{R - n_\parallel^2}{S - n_\parallel^2}\right)^2 \left\{ \frac{\sqrt{\pi}\omega_p^2 \omega^2 \Omega}{2|k_\parallel| v_{th\parallel} c^2} \frac{\lambda}{2k_\perp^2 \Omega + k_\parallel^2(\omega + 2\Omega)} \exp\left[- \left(\frac{\omega - 2\Omega}{k_\parallel v_{th\parallel}} \right)^2 \right] \right\}_i .$$

8.4 Transit Time Damping

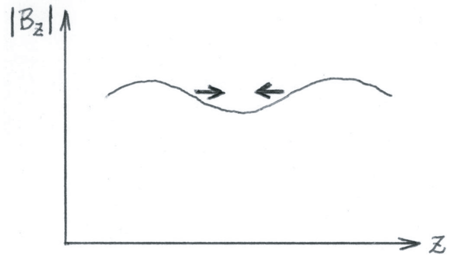
When the wave magnetic field has a gradient along the confining magnetic field, the equation of motion can be written as

$$m\frac{dv_\parallel}{dt} = -\mu \vec{\hat{b}} \cdot \nabla |\vec{B}(\vec{r}, t)|$$

where $\mu = mv_\perp^2/(2B_0)$ is the magnetic moment of a charged particle. Damping described by this equation is called transit time damping, which is the magnetic analog of Landau damping described by

$$m\frac{dv_\parallel}{dt} = -q\vec{\hat{b}} \cdot \nabla \phi.$$

Transit time damping and Landau damping both correspond to the $n = 0$ case, and the two processes are coherent. The net result of the total interaction depends on the relative phases of $E_\parallel$ and $B_\parallel^{(1)}$. It is not correct to calculate them separately and add them together, but cross terms must be considered.



Fg. 2. Schematic showing the mechanism of transit time damping.

As an example, consider the case of compressional Alfvén mode with $|k_\parallel v_{th\parallel}^{(e)}| \simeq \omega \ll \Omega_i$.

$$\begin{pmatrix} \epsilon_\perp - n_\parallel^2 & 0 & n_\parallel n_\perp \\ 0 & \epsilon_\perp + i\Delta\epsilon_\perp - n^2 & i(\epsilon_m + i\Delta\epsilon_m) \\ n_\parallel n_\perp & -i(\epsilon_m + i\Delta\epsilon_m) & \epsilon_\parallel + i\Delta\epsilon_\parallel - n_\perp^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

where

$$\epsilon_\perp = S \simeq 1 + \frac{\rho}{\epsilon B^2}, \quad \Delta\epsilon_\perp = 2\lambda \frac{T_\perp}{T_\parallel}\delta$$

$$\epsilon_\parallel = -\left[\left(\frac{\omega_p^2}{k_\parallel v_{th\parallel}}\right)^2 \Re Z_0'\right]_e, \quad \Delta\epsilon_\parallel = 2\zeta_0^2\delta$$

$$\epsilon_m = -\frac{1}{2}\frac{n_\perp}{n_\parallel}\left[\frac{\omega_p^2}{\omega\Omega}\frac{T_\perp}{T_\parallel}\Re Z_0'\right]_e, \quad \Delta\epsilon_m = \frac{n_\perp}{n_\parallel}\frac{\omega}{\Omega_e}\frac{T_\perp}{T_\parallel}\delta$$

$$\delta = \left[\frac{\omega_p^2}{\omega}\frac{\sqrt{\pi}}{|k_\parallel|v_{th\parallel}}e^{-\zeta_0^2}\right]_e, \quad \zeta_0 = \frac{\omega}{k_\parallel v_{th\parallel}}.$$

Damping is described by the anti-Hermitian parts of the wave equation which occur only in the lower right 2×2 part of the matrix, so the relevant wave equation is

$$\begin{pmatrix} \epsilon_\perp + i\Delta\epsilon_\perp - n^2 & i(\epsilon_m + i\Delta\epsilon_m) \\ -i(\epsilon_m + i\Delta\epsilon_m) & \epsilon_\parallel + i\Delta\epsilon_\parallel - n_\perp^2 \end{pmatrix}\begin{pmatrix} E_y \\ E_z \end{pmatrix} = 0.$$

The real part of the dispersion relation gives

$$(\epsilon_\perp - n^2)\epsilon_\parallel - \epsilon_m^2 \simeq 0$$

and the imaginary part gives

$$\omega_i\frac{\partial}{\partial\omega}\left[\epsilon_\parallel(\epsilon_\perp - n^2) - \epsilon_m^2\right] + \epsilon_\parallel\Delta\epsilon_\perp + \frac{\epsilon_m^2}{\epsilon_\parallel}\Delta\epsilon_\parallel - 2\epsilon_m\Delta\epsilon_m \simeq 0.$$

The last three terms represent transit time damping, Landau damping, and cross terms, respectively. In the case $T_\parallel = T_\perp$, the coherence between $B_z^{(1)}$ (or equivalently E_y) and E_z is given by

$$\frac{E_y}{E_z} \simeq -i\frac{\epsilon_\parallel + i\Delta\epsilon_\parallel}{\epsilon_m + i\Delta\epsilon_m} = -i\frac{2\omega\Omega_e}{k_\perp k_\parallel v_{th\parallel}^2}.$$

8.5 Power Absorption

Under the assumption that $|\omega_i| \ll |\omega_r|$, power absorption can be calculated once the electric field $\vec{E}$ is given,

$$P = \vec{E}\cdot\vec{j} = \sum_s q_s n_s \vec{E}\cdot <\vec{v}>_s .$$

In complex notation,

$$\begin{aligned} P = \sum_s P_s &= \sum_s \frac{n_s q_s}{4}\left(\vec{E}^*\cdot <\vec{v}>_s + \vec{E}\cdot <\vec{v}>_s^*\right) \\ &= -\frac{i\omega\epsilon_0}{4}\sum_s \vec{E}^*\cdot\left(\overleftrightarrow{\chi}_s - \overleftrightarrow{\chi}_s^*\right)\cdot\vec{E} \\ &= \frac{\omega\epsilon_0}{4}\sum_s \vec{E}^*\cdot\overleftrightarrow{\chi}_a\cdot\vec{E}. \end{aligned}$$

For the case of Landau damping with $\vec{E} = \vec{z}E$,

$$P_s = \left\{ \frac{nq^2}{k_\parallel^2 T_\parallel \epsilon_0} e^{-\lambda} I_0(\lambda) \frac{\epsilon_0 |E_\parallel|^2}{2} \frac{\sqrt{\pi}\omega(\omega - k_\parallel V)}{|k_\parallel| v_{th\parallel}} \exp\left[-\left(\frac{\omega - k_\parallel V}{k_\parallel v_{th\parallel}} \right)^2 \right] \right\}_s .$$

For the case of transit time damping with $\vec{E} = \vec{x}E_x + \vec{y}E_y$, $E_\parallel = 0$,

$$P_s = \left\{ \beta_\perp \frac{T_\perp}{T_\parallel} e^{-\lambda} \left[I_0(\lambda) - I_0'(\lambda) \right] \frac{|B_\parallel^{(1)}|^2}{2\mu_0} \frac{\sqrt{\pi}\omega(\omega - k_\parallel V)}{|k_\parallel| v_{th\parallel}} \exp\left[-\left(\frac{\omega - k_\parallel V}{k_\parallel v_{th\parallel}} \right)^2 \right] \right\}_s .$$

If $B_\parallel^{(1)} \neq 0$ and $E_\parallel \neq 0$, Landau damping and transit time damping are coherent, and cross terms ($\sim E_\parallel^* B_\parallel^{(1)}$) must be included.

For the case of cyclotron damping with $E_\parallel = 0$,

$$\begin{aligned} P_s &= \frac{\epsilon_0 \omega_{ps}^2}{2} \sum_{n=-\infty}^{\infty} \left\{ e^{-\lambda} \left[\frac{n^2 I_n}{\lambda} E_x^* E_x - in(I_n - I_n')(E_x^* E_y - E_y^* E_x) \right.\right. \\ &\qquad \left.\left. + \left(\frac{n^2 I_n}{\lambda} + 2\lambda I_n - 2\lambda I_n' \right) E_y^* E_y \right] \right\}_s \Im A_n^{(s)} . \end{aligned}$$

In the limit $\lambda \ll 1$

$$P_s \simeq \frac{\epsilon_0 \omega_{ps}^2}{2} \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \frac{\lambda}{2}^{n-1} \left(|E^+|^2 \Im A_n + |E^-|^2 \Im A_{-n} \right)_s ,$$

and for shifted Maxwellian

$$\Im A_n = \sqrt{\pi} \frac{(\omega - k_\parallel V - n\Omega) T_\perp + n\Omega T_\parallel}{|k_\parallel| v_{th\parallel} \omega T_\parallel} \exp\left[-\left(\frac{\omega - k_\parallel V - n\Omega}{k_\parallel v_{th\parallel}} \right)^2 \right] .$$