7 Susceptibilities for a Hot Magnetized Plasma

7.1 Physical Picture of Cyclotron Damping

Consider a beam of particles travelling along a uniform magnetic field $\vec{B}_0 = B_0\vec{\hat{z}}$ at velocity $\vec{v}_0 = V\vec{\hat{z}}$, with no velocity spread (zero temperature). Assume a wave travelling along the magnetic field $\vec{k} = k\vec{\hat{z}}$ ($k = k_\parallel$) with electric field perpendicular to the magnetic field ($E_\parallel = 0$). The linearized equation of motion is

$$m\frac{\partial \vec{v}}{\partial t} + mV\frac{\partial \vec{v}}{\partial z} = q\left(\vec{E}_1 + \vec{v} \times \vec{\hat{z}}B_0 + V\vec{\hat{z}} \times \vec{B}_1\right).$$

The last term can be rewritten using $\vec{B}_{1\perp} = (k/\omega)\vec{\hat{z}}\times\vec{E}_1$ from Maxwell equation. Using the rotating coordinate system

$$v^\pm = \frac{v_x \pm iv_y}{2}, \quad E^\pm = \frac{E_x \pm iE_y}{2}$$

and the initial condition $\vec{v} = 0$ at $t = 0$, the solution can be expressed as

$$v^\pm(z,t) = \frac{iqE^\pm(\omega - kV)e^{ikz-i\omega t}}{m\omega}\frac{1 - e^{i\omega t - ikVt \mp i\Omega t}}{\omega - kV \mp \Omega}$$

where $\Omega = qB_0/m$. It can be seen that $v^\pm$ grows linearly with time (particle is accelerated in the perpendicular direction by the wave) in the limit $\omega - kV = \pm\Omega$, *i.e.*, when the particle feels the wave electric field oscillating at the cyclotron frequency. This is called the cyclotron resonance.

When there is distibution in velocity, averaging over velocity distribution $N(V)$ gives

$$< \vec{v}_\perp(z,t) >= \Re\left\{\frac{iqe^{ikz-i\omega t}}{2m}\left[(c^+ + c^-)\vec{E}_\perp + i(c^+ - c^-)\vec{E}_\perp \times \vec{\hat{z}}\right]\right\},$$

where

$$c^\pm(t) = \alpha^\pm(t) - i\beta^\pm(t)$$

$$\alpha^\pm(t) = \int_{-\infty}^{\infty} dV\,\frac{N(V)(1 - kV/\omega)[1 - \cos(\omega t - kVt \mp \Omega t)]}{\omega - kV \mp \Omega}$$

$$\beta^\pm(t) = \int_{-\infty}^{\infty} dV\,\frac{N(V)(1 - kV/\omega)\sin(\omega t - kVt \mp \Omega t)}{\omega - kV \mp \Omega}$$

and

$$\int_{-\infty}^{\infty} N(V)dV = 1.$$

For large values of t, these integrals approach asymptotic values independent of t,

$$\alpha^\pm \to P\int_{-\infty}^{\infty} dV\,\frac{N(V)(1 - kV/\omega)}{\omega - kV \mp \Omega}$$

$$\beta^\pm \to \pm\frac{\pi\Omega}{\omega|k|}N\left(\frac{\omega \mp \Omega}{k}\right).$$

Power absorption by the plasma due to cyclotron damping is given by

$$\Re\Big(q\vec{E}\Big) \cdot \Re(< \vec{v}_\perp >) = \frac{q^2}{m}\left(\beta^+|E^+|^2 + \beta^-|E^-|^2\right).$$

7.2 Solution of Vlasov Equation

First consider solution of Vlasov Equation in Lagrangian coordinates, following the 0th order trajectory of particles, $\vec{r}(t)$. The time derivative of the distribution function along the 0th order trajectory is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} + \frac{\partial f}{\partial \vec{p}} \cdot \frac{d\vec{p}}{dt}$$

where $\vec{p} = m\vec{v}$, $m = m_0/\sqrt{1 - v^2/c^2}$, $d\vec{r}/dt = \vec{v}$, $d\vec{p}/dt = q_s\vec{v} \times \vec{B}_0$, since in 0th order $\vec{E} = 0$ and $\vec{B} = \vec{B}_0$.

The 0th order Vlasov equation can be written as

$$\left(\frac{df_{s0}}{dt}\right)_0 = 0$$

where the time derivative is taken along the 0th order particle trajectory. The solution can be given in the form $f_{s0} = f_{s0}(p_\perp, p_\|)$ which doen not depend on $\vec{r}$ or t.

The 1st order Vlasov equation can be written as

$$\left(\frac{df_{s1}}{dt}\right)_0 = -q_s \left(\vec{E}_1 + \vec{v} \times \vec{B}_1\right) \cdot \frac{\partial f_{s0}}{\partial \vec{p}}$$

which can be solved to give

$$f_{s1}(\vec{r}, \vec{p}, t) = -q_s \int_{-\infty}^{t} dt' \left[\vec{E}_1(\vec{r}', t') + \vec{v}' \times \vec{B}_1(\vec{r}', t')\right] \cdot \frac{\partial f_{s0}(\vec{p}')}{\partial \vec{p}'}$$

where the integral is taken along the 0th order particle trajectory.

7.3 Transformation to Eulerian Coordinates

Assume the 1st order electric field to be in the form

$$\vec{E}_1(\vec{r}', t') = \vec{E} \exp(i\vec{k} \cdot \vec{r}' - i\omega t').$$

Substituting $\vec{B}_{1\perp} = (k/\omega)\vec{\hat{z}} \times \vec{E}_1$ and rewriting

$$\vec{v} \times \frac{\vec{k} \times \vec{E}}{\omega} = (\vec{v} \cdot \vec{E})\frac{\vec{k}}{\omega} - \left(\vec{v} \cdot \frac{\vec{k}}{\omega}\right)\vec{E}$$

the 1st order solution can be expressed as

$$f_1(\vec{r}, \vec{p}, t) = -q_s \int_{-\infty}^{t} dt'\, e^{i\vec{k}\cdot\vec{r}' - i\omega t'}\, \vec{E} \cdot \left[\overleftrightarrow{1}\left(1 - \frac{\vec{v}' \cdot \vec{k}}{\omega}\right) + \frac{\vec{v}'\vec{k}}{\omega}\right] \cdot \frac{\partial f_0(\vec{p}')}{\partial \vec{p}'}.$$

Now consider transforming from Lagrangian coordinates along the 0th order trajectory $\vec{r}'$, $\vec{p}'$, t' to Eulerian coordinates $\vec{r}$, $\vec{p}$, t in a stationary frame. Defining

$\tau = t - t'$, and writing $v_x = v_\perp \cos\phi$, $v_y = v_\perp \sin\phi$, the relationship between the two coordinate systems can be expressed as

$$\begin{aligned} v'_x &= v_\perp \cos(\phi + \Omega\tau) \\ v'_y &= v_\perp \sin(\phi + \Omega\tau) \\ v'_z &= v_\parallel \end{aligned}$$

and integrating once

$$\begin{aligned} x' &= x - \frac{v_\perp}{\Omega}\left[\sin(\phi + \Omega\tau) - \sin\phi\right] \\ y' &= y + \frac{v_\perp}{\Omega}\left[\cos(\phi + \Omega\tau) - \cos\phi\right] \\ z' &= z - v_\parallel \tau \end{aligned}$$

Further taking $k_x = k_\perp \cos\theta$, $k_y = k_\perp \sin\theta$, and $\vec{k}\cdot\vec{r}' - \omega t' = \vec{k}\cdot\vec{r} - \omega t + \beta$ where

$$\beta = -\frac{k_\perp v_\perp}{\Omega}\left[\sin(\phi - \theta + \Omega\tau) - \sin(\phi - \theta)\right] + (\omega - k_\parallel v_\parallel)\tau$$

the 1st order distribution function can be expressed as

$$\begin{aligned} f_1(\vec{r}, \vec{p}, t) &= -q e^{i\vec{k}\cdot\vec{r} - i\omega t} \int_0^\infty d\tau\, e^{i\beta} \Big\{ E_x U \cos(\phi + \Omega\tau) + E_y U \sin(\phi + \Omega\tau) \\ &\quad + E_z \left[\frac{\partial f_0}{\partial p_\parallel} - V \cos(\phi - \theta + \Omega\tau)\right] \Big\} \end{aligned}$$

where

$$\begin{aligned} U &= \frac{\partial f_0}{\partial p_\perp} + \frac{k_\parallel}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp}\right) \\ V &= \frac{k_\perp}{\omega}\left(v_\perp \frac{\partial f_0}{\partial p_\parallel} - v_\parallel \frac{\partial f_0}{\partial p_\perp}\right). \end{aligned}$$