6 Landau Damping

6.1 Physical Picture of Landau Damping

Consider a 1-dimensional electrostatic (longitudinal) wave with $\vec{k} \parallel \vec{E}$ in the absence of magnetic field. Taking $\vec{v} = \vec{z}v$ and $\vec{E} = \vec{z}E\cos(kz - \omega t)$, the single-particle equation of motion can be written as

$$m\frac{dv}{dt} = qE\cos(kz - \omega t).$$

The 0th order solution (for $E = 0$) $z = v_0 t + z_0$ can be substituted into the 1st order equation to give

$$m\frac{dv_1}{dt} = qE\cos(kz_0 + kv_0 t - \omega t).$$

This equation is solved as an initial value problem with the initial condition $v_1 = 0$ at $t = 0$. The solution is given by

$$v_1 = \frac{qE}{m}\frac{\sin(kz_0 + kv_0 t - \omega t) - \sin(kz_0)}{kv_0 - \omega}.$$

The time rate of change of the kinetic energy, averaged over initial positions z_0 is

$$\left\langle \frac{d}{dt}\frac{mv^2}{2} \right\rangle_{z_0} = \frac{q^2E^2}{2m}\left[-\frac{\omega\sin(\alpha t)}{\alpha^2} + t\cos(\alpha t) + \frac{\omega t\cos(\alpha t)}{\alpha} \right]$$

where $\alpha = kv_0 - \omega$. This is further averaged over the distribution of initial velocities v_0

$$f(v_0) = f\left(\frac{\alpha + \omega}{k}\right) = g(\alpha)$$

to give

$$\left\langle \frac{d}{dt}\frac{mv^2}{2} \right\rangle_{z_0,\, v_0} = -\frac{\omega q^2 E^2}{2m|k|}\mathrm{P}\int_{-\infty}^{\infty} d\alpha\, \frac{g(\alpha)\sin(\alpha t)}{\alpha^2}.$$

Expanding in the vicinity of $\alpha = 0$

$$g(\alpha) = g(0) + \alpha g'(0) + \frac{\alpha^2}{2}g'' + \cdots$$

gives

$$\left\langle \frac{d}{dt}\frac{mv^2}{2} \right\rangle_{z_0,\, v_0} \simeq -\frac{\pi\omega q^2 E^2}{2mk|k|}\left[\frac{df(v_0)}{dv_0}\right]_{v_0 = \frac{\omega}{k}}.$$

This expression signifies that resonant particles with velocity close to the wave phase velocity determine absorption of wave power by particles.

6.2 A Simple Kinetic Model

Consider a 1-dimensional oscillation along the magnetic field (or in the absence of magnetic field). Vlasov equation can be written as

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} + \frac{q}{m}E(z,t)\frac{\partial f}{\partial v} = 0.$$

The first order equation is

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial z} = -\frac{q}{m}E(z,t)\frac{df_0(v)}{dv}.$$

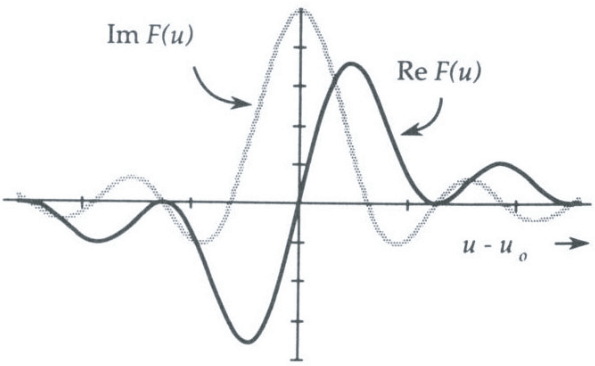
The left hand side is called the convective derivative, and signifies time derivative along the particle trajectory. Taking $E(z,t) = \Re[E_1 \exp(ikz - i\omega t)]$, the solution can be expressed as

$$f_1(z,v,t) = g_1(z-vt,v) - \Re\left[\frac{iqE_1}{m}\frac{df_0(v)}{dv}e^{ikz-i\omega t}\frac{1-e^{i(\omega-kv)(t-t_0)}}{\omega-kv}\right],$$

where $g_1(z-vt,v)$ is the homogeneous solution (solution for $E=0$), and is chosen to satisfy the initial condition. The last term

$$\begin{aligned} F(u) &= -\frac{1-e^{i(u_0-u)\tau}}{u_0-u} \\ &= -\frac{1-e^{i(\omega-kv)(t-t_0)}}{\omega-kv} \end{aligned}$$

is shown in Fig. 1.

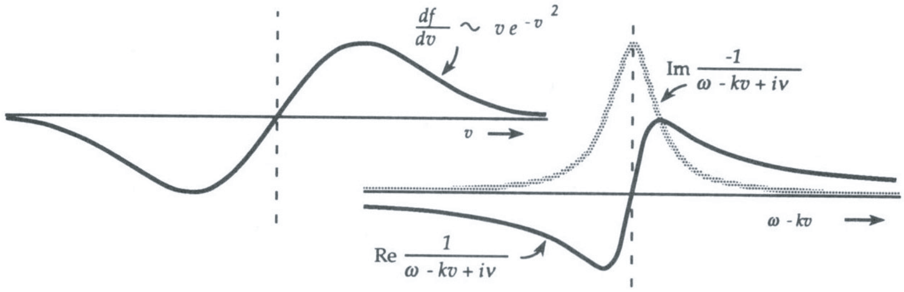


Fg. 1. Real and imaginary parts of the resonance term $F(u)$.

It was pointed out in Chap. 3 that to satisfy causality the integration contour on the complex ω plane must be taken above the singularity that exists on the real ω axis. This is mathematically equivalent to shifting the singularity slightly to the negative imaginary side of the real axis. This can be accomplished by introducing randomization by collisions. The probability of not suffering collisions since $t = t_0$ is given by $\exp[-\nu(t-t_0)]$, where ν is the collision frequency. Multiplying this probability and averaging over all particles reaching z and v at time t,

$$\langle f_1(z,v,t)\rangle = -\Re\left[\frac{iqE_1}{m}\frac{df_0(v)}{dv}e^{ikz-i\omega t}\frac{1}{\omega-kv+i\nu}\right].$$

For a nearly Maxwellian distribution function, df_0/dv has a velocity width of order v_{th}, whereas $1/(\omega-kv+i\nu)$ has a width of ν/k, as illustrated in Fig. 2.



Fg. 2. The functions $\dfrac{df_0}{dv}$ and $\dfrac{1}{\omega - kv + i\nu}$.

6.3 Validity Conditions for Landau Damping

The bottom of a sinusoidal potential well can be approximated as

$$q\phi\left[1-\cos(kx)\right] \simeq \frac{q\phi}{2}k^2x^2.$$

A charged particle oscillates harmonically in this potential with period

$$\tau_{\rm osc} = \frac{1}{\omega_{\rm osc}} = \sqrt{\frac{m}{qkE}}.$$

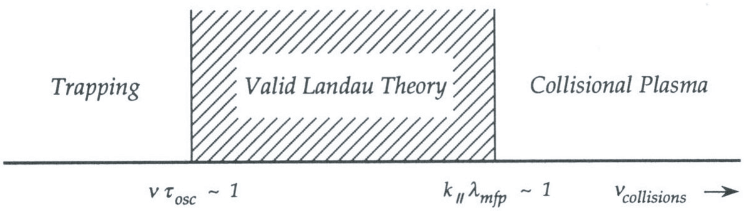
The following conditions must be satisfied for valid Landau damping.

1. $|\omega_i \tau_{\rm osc}| > 1$ Significant growth (or damping) must occur before $v_\parallel$ changes substantially (which occurs in an oscillation time in the potential well, $\tau_{\rm osc}$), since linear theory assumes $v_\parallel = $ const.

2. $\nu_{\rm coll}\tau_{\rm osc} > 1$ The collision time $\tau_{\rm coll}$ must be shorter than $\tau_{\rm osc}$ to ensure the assumption $v_\parallel = $ const.

3. $k_\parallel \lambda_{\rm mfp} > 1$ ($v_{th} > \nu_{\rm coll}/k_\parallel$) Mean free path must be longer than a wavelength for particles to recognize the presence of a wave.

The valid range of collision frequency is shown in Fig. 3.



Fg. 3. The valid range of collision frequency for Landau damping.

6.4 ES Waves in a Maxwellian Unmagnetized Plasma

Assume a drifting Maxwellian plasma, given by

$$f_s(v) = \frac{n_s}{\sqrt{\pi} v_{ths}} \exp\left[-\frac{(v-V_s)^2}{v_{ths}^2} \right],$$

where $v_{ths}^2 = 2T_s/m_s$. For real k, $\Im\omega > 0$ and defining $\tau = t - t'$

$$f_1(\omega, k, v) = \frac{nqE(\omega,k)}{\sqrt{\pi} m v_{th}} \frac{d}{dV} \int_0^\infty d\tau e^{i(\omega - kv)\tau} e^{\frac{-(v-V)^2}{v_{th}^2}}.$$

It is useful to define v^p moments of the distribution function,

$$Z_p(\omega, v_{th\|}, k_\|, V, n\Omega) = \frac{ik_\|}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv\, v^p \int_0^\infty d\tau\, e^{i(\omega - n\Omega - k_\| v)\tau} e^{-\frac{(v-V)^2}{v_{th\|}^2}}.$$

In particular, for $p = 0$

$$Z_0(\zeta_n) = i\sqrt{\pi}\,\mathrm{sgn}(k_\|) e^{-\zeta_n^2} - 2S(\zeta_n)$$

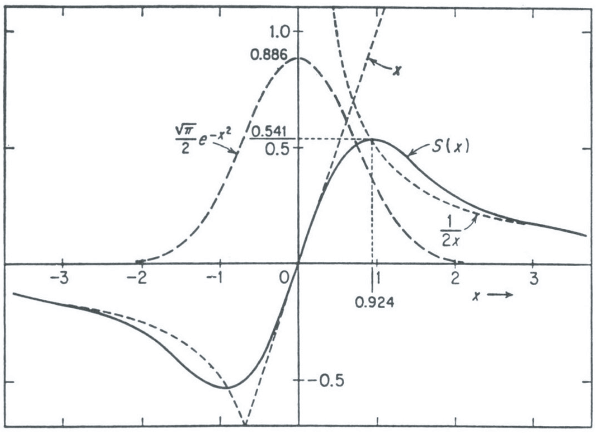
where

$$\zeta_n = \frac{\omega - k_\| V - n\Omega}{k_\| v_{th\|}}$$

and

$$S(\zeta) = e^{-\zeta^2} \int_0^\zeta dz\, e^{z^2} = -S(-\zeta).$$

The functions $S(\zeta)$ and $(\sqrt{\pi}/2)\exp(-\zeta^2)$ are illustrated in Fig. 4.



Fg. 4. The functions $S(x)$ and $\frac{\sqrt{\pi}}{2} e^{-x^2}$ which describe the real and imaginary parts of the plasma dispersion function.

The dispersion relation for a 1-D electrostatic wave in the absence of magnetic field can be derived from Poisson's equation

$$ikE(\omega, k) = \sum_s \frac{q_s}{\epsilon_0} \int_{-\infty}^{\infty} f_{s1}(v, \omega, k) dv$$

as

$$k^2 = \sum_s \frac{1}{2\lambda_{ds}^2} Z_0'(\zeta^{(s)})$$

where

$$\lambda_{ds}^2 = \frac{n_s q_s^2}{\epsilon_0 T_s}$$

and

$$Z_0'(\zeta) = -2\left[1 + \zeta Z_0(\zeta)\right].$$

A closely related function is called the plasma dispersion function, which is defined as

$$Z(\zeta) = i \int_0^\infty dz \, \exp\left(i\zeta z - \frac{z^2}{4}\right).$$

The functions Z_0 and Z are related to each other as

$$Z_0(\zeta) = \begin{cases} Z(\zeta) & \text{for } k_\parallel > 0 \\ -Z(-\zeta) & \text{for } k_\parallel < 0 \end{cases}$$