4 Energy Flow and Accessibility

4.1 Wave Energy and Energy Transport

The dielectric property of the plasma is described by the dielectric tensor $\overleftrightarrow{\epsilon}$ $(\vec{r}, t, \omega, \vec{k})$, where $\vec{r}$ and t dependences represent slow variations in space and time compared to the wavelength and the wave period. The wave electric field is expressed as $\vec{E} = \Re\left(\vec{E}_1 e^{-i\phi}\right)$, where ϕ is the eikonal

$$\phi(\vec{r}, t) = -\int_{-\infty}^{\vec{r}} \vec{k}(\vec{r}') \cdot d\vec{r}' + \int_{-\infty}^{t} \omega(t')dt'.$$

The product of two vector quantities $\vec{A}^{(1)}$ and $\vec{B}^{(1)}$ are given by

$$\begin{aligned}\vec{A}^{(1)}\vec{B}^{(1)} &= \frac{1}{2}\left(\vec{A}_1 e^{-i\phi} + \vec{A}_1^* e^{i\phi^*}\right)\frac{1}{2}\left(\vec{B}_1 e^{-i\phi} + \vec{B}_1^* e^{i\phi^*}\right) \\ &= \frac{1}{4}\left[\vec{A}_1\vec{B}_1 e^{-2i\phi} + \vec{A}_1^*\vec{B}_1^* e^{2i\phi^*} + \vec{A}_1\vec{B}_1^* e^{-i(\phi-\phi^*)} + \vec{A}_1^*\vec{B}_1 e^{i(\phi^*-\phi)}\right].\end{aligned}$$

Using $\phi = \phi_\mathrm{r} + i\phi_\mathrm{i}$, and averaging over a few wave periods gives

$$\left\langle \vec{A}^{(1)}\vec{B}^{(1)}\right\rangle = \frac{1}{4}\left(\vec{A}_1\vec{B}_1^* + \vec{A}_1^*\vec{B}_1\right) e^{2\phi_\mathrm{i}(\vec{r},t)}.$$

Poynting's theorem can be derived from Maxwell equations.

$$\nabla \cdot \left(\vec{E} \times \vec{H}\right) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = 0.$$

The first term represents divergence of electromagnetic energy flux $\nabla \cdot \vec{P}$ and the second and third terms represent time derivative of the energy density $\partial W/\partial t$, where

$$\vec{P} = \frac{1}{4\mu_0}\left(\vec{E}_1^* \times \vec{B}_1 + \vec{E}_1 \times \vec{B}_1^*\right) e^{2\phi_\mathrm{i}(\vec{r},t)}$$

$$\begin{aligned}\frac{\partial W}{\partial t} &= \frac{1}{4}\left[2\omega_\mathrm{i}\frac{\vec{B}_1^* \cdot \vec{B}_1}{\mu_0} + \omega_\mathrm{i}\epsilon_0\vec{E}_1^* \cdot (\overleftrightarrow{\epsilon} + \overleftrightarrow{\epsilon}^\dagger) \cdot \vec{E}_1\right. \\ &\qquad \left. + \omega_\mathrm{r}\epsilon_0\vec{E}_1^* \cdot (-i\overleftrightarrow{\epsilon} + i\overleftrightarrow{\epsilon}^\dagger) \cdot \vec{E}_1\right] e^{2\phi_\mathrm{i}(\vec{r},t)}\end{aligned}$$

The wave energy consists of electromagnetic energy and acoustic energy, *i.e.*, kinetic energy of coherent particle motion. The dielectric constant can be decomposed into Hermitian and anti-Hermitian parts

$$\begin{aligned}\overleftrightarrow{\epsilon} &= \overleftrightarrow{\epsilon}_h + i\overleftrightarrow{\epsilon}_a \\ \overleftrightarrow{\epsilon}^\dagger &= \overleftrightarrow{\epsilon}_h - i\overleftrightarrow{\epsilon}_a\end{aligned}$$

where the dagger denotes Hermitian conjugate, *i.e.*, $\epsilon_{ij}^\dagger = \epsilon_{ji}^*$. For $\omega_i \neq 0$ or $\vec{k}_i \neq 0$,

$$\overleftrightarrow{\epsilon}_h(\omega_r + i\omega_i, \vec{k}_r + i\vec{k}_i) = \overleftrightarrow{\epsilon}_h(\omega_r, \vec{k}_r) + \left[i\omega_i\frac{\partial}{\partial\omega}\overleftrightarrow{\epsilon}_h + i\vec{k}_i\frac{\partial}{\partial\vec{k}}\overleftrightarrow{\epsilon}_h\right]_{\omega_r,\vec{k}_r} + \cdots$$

$$\frac{\partial W}{\partial t} = \frac{1}{2}\left[\omega_i \frac{\vec{B}_1^* \cdot \vec{B}_1}{\mu_0} + \omega_i \epsilon_0 \vec{E}_1^* \cdot \overleftrightarrow{\epsilon}_h \cdot \vec{E}_1 \right.$$
$$\left. + \omega_r \epsilon_0 \vec{E}_1^* \cdot \left(\overleftrightarrow{\epsilon}_a + \omega_i \frac{\partial}{\partial \omega} \overleftrightarrow{\epsilon}_h + \vec{k}_i \cdot \frac{\partial}{\partial \vec{k}} \overleftrightarrow{\epsilon}_h \right) \cdot \vec{E}_1 \right] e^{2\phi_i(\vec{r},t)}$$

Poynting's theorem becomes

$$\nabla \cdot \vec{P} + \frac{\partial W}{\partial t} = -2\vec{k}_i \cdot \left(\vec{P} + \vec{T} \right) + 2\omega_i W + \left. \frac{\partial W}{\partial t} \right|_{\text{lossy}} = 0$$

where

$$\vec{P} = \frac{1}{4\mu_0} \left(\vec{E}^* \times \vec{B} + \vec{E} \times \vec{B}^* \right)$$

is the flux of electromagnetic energy,

$$\vec{T} = -\frac{\omega \epsilon_0}{4} \vec{E}^* \cdot \frac{\partial}{\partial \vec{k}} \overleftrightarrow{\epsilon}_h \cdot \vec{E}$$

is the flux of acoustic energy,

$$W = \frac{1}{4} \left[\frac{\vec{B}^* \cdot \vec{B}}{\mu_0} + \epsilon_0 \vec{E}^* \cdot \frac{\partial}{\partial \omega} \left(\omega \overleftrightarrow{\epsilon}_h \right) \cdot \vec{E} \right]$$

and

$$\left. \frac{\partial W}{\partial t} \right|_{\text{lossy}} = \frac{\omega_r \epsilon_0}{2} \vec{E}^* \cdot \overleftrightarrow{\epsilon}_a \cdot \vec{E}$$

expresses dissipation (absorption).

4.2 Group Velocity and Ray Tracing

Consider a wave packet

$$\vec{E}(\vec{r}, t) \sim \int d^3\vec{k}\, \vec{E}(\vec{k}) \exp \left[i\vec{k} \cdot \vec{r} - i\omega(\vec{k})t \right].$$

The locus of strongest constructive interference, where the phase of the integrand is stationary, travels at the velocity

$$\vec{v}_g = \frac{d\vec{r}}{dt} = \frac{\partial \omega(\vec{k})}{\partial \vec{k}}.$$

The direction of energy flow (ray direction) is along the group velocity vector. The ray direction is generally different from the direction of the propagation vector $\vec{k}$ in anisotropic media. The group velocity can be decomposed into components parallel and perpendicular to $\vec{k}$

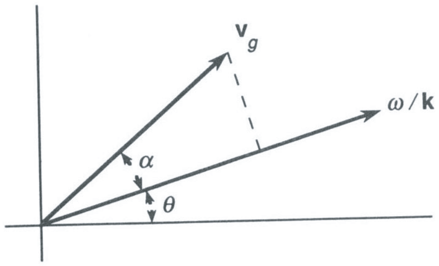
$$\vec{v}_g = \vec{\hat{k}} \frac{\partial \omega}{\partial k} + \vec{\hat{\theta}} \frac{1}{k} \frac{\partial \omega}{\partial \theta},$$

whereas the phase velocity can be expressed as

$$\vec{v}_{ph} = \frac{\omega}{\vec{k}} = \vec{\hat{k}} \frac{\omega}{k}.$$

The angle α between $\vec{v}_g$ and $\vec{k}$ is given by

$$\tan\alpha = \frac{\frac{1}{k}\left(\frac{\partial\omega}{\partial\theta}\right)_k}{\left(\frac{\partial\omega}{\partial k}\right)_\theta} = -\frac{1}{k}\frac{\partial k}{\partial\theta} = -\frac{1}{n}\frac{\partial n}{\partial\theta}.$$



Directions of phase and group velocities.

The dispersion relation can be written in the form

$$g(\vec{r}, t, \vec{k}, \omega) = 0.$$

The following Hamiltonian equations can be derived

$$\begin{aligned}
\frac{d\vec{r}}{d\tau} &= \frac{\partial g}{\partial\vec{k}} \\
\frac{d\vec{k}}{d\tau} &= -\frac{\partial g}{\partial\vec{r}} \\
\frac{dt}{d\tau} &= -\frac{\partial g}{\partial\omega} \\
\frac{d\omega}{d\tau} &= \frac{\partial g}{\partial t}
\end{aligned}$$

where τ is a measure of distance along the ray trajectory. Taking the dispersion relation in the form $\omega = \omega(\vec{r}, t, \vec{k})$ the ray tracing equations can be written in the form

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \frac{\partial\omega}{\partial\vec{k}} = \vec{v}_g \\
\frac{d\vec{k}}{dt} &= -\frac{\partial\omega}{\partial\vec{r}}.
\end{aligned}$$

4.3 Accessibility

In WKB approximation, the variation of plasma parameters (taken to be in the x-direction) is assumed to be sufficiently slow within a wavelength. In this case,

k_y and k_z remain constant, whereas k_x^2 changes slowly with x. In a cold plasma, the dispersion relation is given by $an_x^4 - bn_x^2 + c = 0$, where

$$\begin{aligned} a &= S \\ b &= RL + PS - Pn_\parallel^2 - Sn_\parallel^2 \\ c &= P(RL - 2Sn_\parallel^2 + n_\parallel^4). \end{aligned}$$

The solutions are given by

$$n_x^2 = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

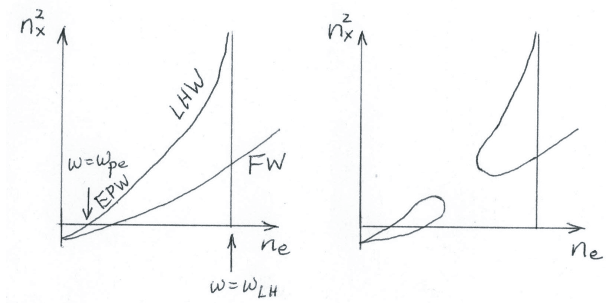
Consider the lower hybrid wave. In the low density region near the edge of the plasma $|D| \ll 1$, $S \simeq 1$, and $RL \simeq 1$, so

$$\left[n_x^2 - P\left(1 - n_\parallel^2\right)\right]\left[n_x^2 - \left(1 - n_\parallel^2\right)\right] = 0.$$

Waves with $n_\parallel^2 > 1$ are used for heating and current drive by Landau damping. In this case, the second root given by $n_x^2 = 1 - n_\parallel^2 < 0$ is evanescent. The first root given by $n_x^2 = P(1 - n_\parallel^2)$ describes a propagating mode (electron plasma wave) for $P < 0$ ($\omega_{pe} > \omega$), but is evanescent for $P > 0$ ($\omega_{pe} < \omega$). In the high density core region, $|P| \gg 1$, $|P| \gg |S|$, so

$$\begin{aligned} a &> 0 \\ b &> 0 \quad \text{if} \quad n_\parallel^2 > \left|\frac{RL}{P}\right| + |S| \\ b^2 - 4ac &> 0 \quad \text{if} \quad n_\parallel^2 > \left(\sqrt{S} + \sqrt{\left|\frac{D^2}{P}\right|}\right)^2. \end{aligned}$$

The second condition is satisfied automatically if the third condition is satisfied.



Accessible (left) and inaccessible (right) cases for the lower hybrid wave.