3 Causality, ES and EM Approximations

3.1 Nonlocal Behavior

$$\vec{D}(\omega, \vec{k}) = \epsilon_0 \overleftrightarrow{\epsilon}(\omega, \vec{k}) \cdot \vec{E}(\omega, \vec{k})$$

is Fourier transformed to give

$$\vec{D}(\vec{r}, t) \sim \int d^3\vec{r}\,' \int dt' \overleftrightarrow{\epsilon}(\vec{r}, t, \vec{r}\,', t') \cdot \vec{E}(\vec{r}\,', t')$$

This means that $\vec{D}$ at $(\vec{r}, t)$ depends on $\vec{E}$ at all $(\vec{r}\,', t')$.

Consider a 1-dimensional case, homogeneous in space and time.

$$\vec{E}(\omega, k) = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-i(kz-\omega t)} \vec{E}(z, t)$$

$$\vec{E}(z, t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{i(kz-\omega t)} \vec{E}(\omega, k)$$

The electric displacement can be expressed as

$$\vec{D}(z, t) = \frac{\epsilon_0}{2\pi} \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dt' \overleftrightarrow{\epsilon}(z - z', t - t') \cdot \vec{E}(z', t').$$

This shows nonlocal response, and $\overleftrightarrow{\epsilon}(z-z', t-t')$ is called a correlation function.

3.2 Causality

In a physical system, response occurs after stimulus.

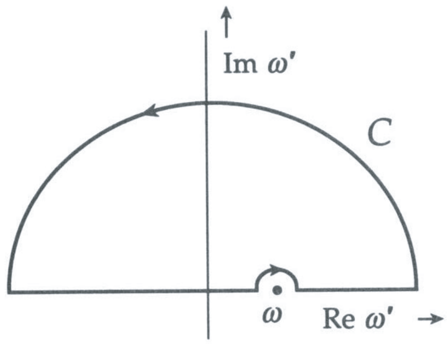
$$\vec{j}_s(\omega) = -i\omega\epsilon_0 \overleftrightarrow{\chi}_s(\omega) \cdot \vec{E}(\omega)$$

If the stimulus is given by $E(t) = E_0\delta(t)$, $E(\omega) = E_0/\sqrt{2\pi}$, so

$$j_s(t) = -i\frac{\epsilon_0 E_0}{2\pi} \int_{-\infty}^{\infty} d\omega\, \omega\chi_s(\omega) e^{-i\omega t}$$

The correct solution must have $j_s(t) = 0$ for $t < 0$. Writing $\omega = \omega_{\mathrm{re}} + i\omega_{\mathrm{im}}$, $\exp(-i\omega t) = \exp(-i\omega_{\mathrm{re}}t + \omega_{\mathrm{im}}t)$. To satisfy causality, $\omega\chi_s(\omega)$ must be analytic (have no singularities) for $\omega_{\mathrm{im}} > 0$ (in the upper half plane). For example, for the left-hand circularly polarized wave $\omega\chi_s(\omega) = -\omega_{ps}^2/(\omega - \Omega_s)$, there is a singularity at $\omega = \Omega_s$. To satisfy causality, the integration contour must pass above the singularity

$$\omega\chi_s = -\lim_{\nu\to 0^+} \frac{\omega_{ps}^2}{\omega + i\nu - \Omega_s} = -\omega_{ps}^2 \left[P\left(\frac{1}{\omega - \Omega_s}\right) - i\pi\delta(\omega - \Omega_s) \right].$$



Integration contour chosen to satisfy causality.

3.3 Electrostatic and Electromagnetic Approximations

Electrostatic approximation

In the electrostatic (ES) approximation, $\vec{E} = -\nabla\phi$, so $\vec{E} \parallel \vec{k}$ (longitudinal mode). Since $\vec{D} = \epsilon_0 \overleftrightarrow{\epsilon} \cdot \vec{E}$, $\nabla \cdot \vec{D} = 0$ yields $\vec{k} \cdot \overleftrightarrow{\epsilon} \cdot \vec{k}\phi = 0$. Electrostatic dispersion relation for uniform medium is given by

$$\hat{n} \cdot \overleftrightarrow{\epsilon} \cdot \hat{n} = 0.$$

The wave equation $\vec{n} \times (\vec{n} \times \vec{E}) + \overleftrightarrow{\epsilon} \cdot \vec{E} = 0$ can be rewritten in terms of electric field components parallel and perpendicular to $\vec{k}$, $\vec{E}_\parallel$ and $\vec{E}_\perp$

$$\left(n^2 \overleftrightarrow{1} - \overleftrightarrow{\epsilon}\right) \cdot \vec{E}_\perp = \overleftrightarrow{\epsilon} \cdot \vec{E}_\parallel.$$

Taking the dot product of this wave equation with the the transverse unit vector $\hat{t}$ yields

$$\frac{E_\perp}{E_\parallel} = \frac{\hat{t} \cdot \overleftrightarrow{\epsilon} \cdot \hat{n}}{n^2 - \hat{t} \cdot \overleftrightarrow{\epsilon} \cdot \hat{t}}.$$

$|E_\perp| \ll |E_\parallel|$ must be satisfied for ES wave. This is satisfied if $n^2 \gg |\epsilon_{ij}|$ is satisfied for any i and j. This means the wavelength is short.

Electromagnetic approximation

In contrast to ES mode, electromagnetic (EM) mode has $|E_\perp| \gg |E_\parallel|$. Taking the dot product of the wave equation with $\hat{t}$ yields the dispersion relation for EM wave

$$n^2 - \hat{t} \cdot \overleftrightarrow{\epsilon} \cdot \hat{t} = 0.$$

Taking the dot product of the wave equation with $\hat{n}$ yields

$$\frac{E_\parallel}{E_\perp} = -\frac{\hat{n} \cdot \overleftrightarrow{\epsilon} \cdot \hat{t}}{\hat{n} \cdot \overleftrightarrow{\epsilon} \cdot \hat{n}}.$$

3.4 Finite Temperature Correction

Correction due to finite electron temperature introduces new modes. The plasma oscillation becomes a propagating mode, the electron plasma wave. New modes which had no analogue in cold plasma, such as the ion acoustic wave and the drift wave can exist at finite electron temperature, even within the framework of fluid theory. Depending on the relative magnitude of the electron thermal velocity $v_{th}^{(e)} \equiv \sqrt{2T_e/m_e}$ and the phase velocity of the wave along the magnetic field $\omega/k_\parallel$, plasma responds differently.

In first order in wave quantities, the continuity equation, the equation of motion, and the equation of state for electrons are given by

$$\frac{\partial n_{e1}}{\partial t} + \nabla \cdot (n_{e0}\vec{v}_{e1}) = 0$$

$$n_{e0} m_e \frac{\partial \vec{v}_{e1}}{\partial t} = -n_{e0} e \left(\vec{E} + \vec{v}_{e1} \times \vec{B}_0 \right) - \hat{z} \frac{\partial}{\partial z} p_{e1}^{(\parallel)}$$

$$\begin{array}{lll} \text{adiabatic} & \left(\dfrac{\omega}{k_\parallel} \gg v_{th}^{(e)} \right) & \dfrac{1}{p_{e0}^{(\parallel)}} \dfrac{dp_{e1}^{(\parallel)}}{dt} = \dfrac{\gamma}{n_{e0}} \dfrac{dn_{e1}}{dt} \\ \text{isothermal} & \left(\dfrac{\omega}{k_\parallel} \ll v_{th}^{(e)} \right) & p_{e1}^{(\parallel)} = n_{e1} T_{e0}^{(\parallel)}. \end{array}$$

In the adiabatic case $p_e \propto n_e^\gamma$. In the isothermal case $p_e = n_e T_e$, and $T_{e1}^{(\parallel)} = 0$ is assumed. Kinetic theory, which takes the distribution function into account, is required in the intermediate case when $\omega/k_\parallel$ is comparable to $v_{th}^{(e)}$.

In the low frequency electrostatic approximation ($\omega \ll |\Omega_e|$ and $\vec{E} = -\nabla\phi$), $\vec{v}_{e\perp} = \vec{E} \times \vec{B}_0/B_0^2 = -\nabla\phi \times \vec{B}_0/B_0^2$, and $\nabla \cdot \vec{v}_{e\perp} = 0$. The electron continuity equation becomes

$$-i\omega n_{e1} + \frac{ik_x\phi}{B_0}\frac{dn_{e0}}{dy} + in_{e0}k_z v_{ez1} = 0.$$

The z component of the electron velocity v_{ez1} can be derived from the equation of motion using the adiabatic and isothermal equations of motion. In the adiabatic and isothermal limits, the zz component of the electron susceptibility is given by

$$\begin{array}{lll} \text{adiabatic} & \chi_{zz}^{(e)} & = \dfrac{\omega_{pe}^2}{-\omega^2 + \gamma k_\parallel^2 T_\parallel^{(e)}/m_e} \left[1 - \dfrac{k_x}{\omega e B} \dfrac{d\left(n_e T_\parallel^{(e)}\right)}{n_e dy} \right] \\ \text{isothermal} & \chi_{zz}^{(e)} & = \dfrac{\omega_{pe}^2}{-\omega^2 + k_\parallel^2 T_\parallel^{(e)}/m_e} \left[1 - \dfrac{k_x T_\parallel^{(e)}}{\omega e B} \dfrac{dn_e}{n_e dy} \right] \end{array}$$

where in this case the degree of freedom is $n = 1$, and therefore the ratio of specific heats of $\gamma = (n+2)/n = 3$ is used.

Electron plasma wave

In the cold plasma approximation, $\epsilon_{zz} = 0$ gave the plasma oscillation. In the warm plasma approximation (finite electron temperature, but in the adiabatic limit), and in the absence of density gradients, $\epsilon_{zz} = 0$ gives the dispersion relation for the electron plasma wave

$$\omega^2 = \omega_{pe}^2 \left(1 + \frac{3k_\parallel^2 T_\parallel^{(e)}}{\omega^2 m_e} \right).$$

Ion acoustic wave

In the isothermal limit for electrons, but adiabatic limit for ions, $v_{th}^{(i)} \ll |\omega/k_\parallel| \ll v_{th}^{(e)}$,

$$\epsilon_{zz} = 1 - \frac{\omega_{pi}^2}{\omega^2 - 3k_\parallel^2 T_\parallel^{(i)}/m_i} + \frac{\omega_{pe}^2 m_e}{k_\parallel^2 T_\parallel^{(e)}} = 0.$$

This leads to the dispersion relation for the ion acoustic wave

$$\omega^2 = k_\parallel^2 \frac{3T_\parallel^{(i)} + ZT_\parallel^{(e)}}{m_i}.$$

Drift wave

At low frequencies $\omega \ll \Omega_i$ and in the presence of density gradient, the drift wave can exist. In the isothermal limit, the dispersion relation is given by

$$\omega = \omega^* = \frac{k_x T_\parallel^{(e)}}{eB} \frac{1}{n_e} \frac{dn_e}{dy}.$$