2 Waves in a Cold Uniform Plasma

2.1 Parallel and Perpendicular Propagation

Parallel propagation ($\theta = 0$, $n = n_\parallel$)

$$
\begin{aligned}
P = 0 \qquad & \omega^2 = \omega_{pe}^2 \\
n_\parallel^2 = R \qquad & \frac{c^2 k_\parallel^2}{\omega^2} = \frac{\omega^2 + \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2}{(\omega + \Omega_i)(\omega + \Omega_e)} \\
n_\parallel^2 = L \qquad & \frac{c^2 k_\parallel^2}{\omega^2} = \frac{\omega^2 - \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2}{(\omega - \Omega_i)(\omega - \Omega_e)}
\end{aligned}
$$

Perpendicular propagation ($\theta = \pi/2$, $n = n_\perp$)

$$
\begin{aligned}
n_\perp^2 = P \qquad & c^2 k_\perp^2 = \omega^2 - \omega_{pe}^2 \\
n_\perp^2 = \frac{RL}{S} \qquad & \frac{c^2 k_\perp^2}{\omega^2} = \frac{(\omega^2 + \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2)(\omega^2 - \omega\Omega_e + \Omega_e\Omega_i - \omega_{pe}^2)}{(\omega^2 - \omega_{\mathrm{LH}}^2)(\omega^2 - \omega_{\mathrm{UH}}^2)}
\end{aligned}
$$

where ω_{LH} and ω_{UH} are lower and upper hybrid frequencies

$$
\begin{aligned}
\frac{1}{\omega_{\mathrm{LH}}^2} &= \frac{1}{\Omega_i^2 + \omega_{pi}^2} + \frac{1}{|\Omega_i\Omega_e|} \\
\omega_{\mathrm{UH}}^2 &= \Omega_e^2 + \omega_{pe}^2 .
\end{aligned}
$$

2.2 Alfvén Waves

At sufficiently low frequencies $\omega \ll \Omega_i$, the equation of motion $m_s d\vec{v}_s/dt = q_s(\vec{E} + \vec{v}_s \times \vec{B})$ simply becomes $\vec{E} + \vec{v}_s \times \vec{B} = 0$, which means that the plasma moves at the $\vec{E} \times \vec{B}$ velocity (with the magnetic field lines). This is called the MHD approximation.

$$
\vec{v} = \frac{\vec{E} \times \vec{B}}{B^2}.
$$

In this approximation, $R \simeq L \simeq S \simeq 1 - \omega_{pe}^2/(\Omega_e\Omega_i) \equiv 1 + \gamma$ and $D \simeq 0$, so the dielectric tensor becomes diagonal

$$
\overleftrightarrow{\epsilon} = \begin{pmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}
$$

where $\epsilon_\perp = 1 + \gamma$ and $\epsilon_\parallel = P$. Since $|P|/(1+\gamma) \simeq |\Omega_i\Omega_e|/\omega^2 \gg 1$ (except for very low densities $\omega_{pi}^2 \ll \Omega_i^2$) and $\left|P\cos^2\theta\right| \gg \left|S\sin^2\theta\right|$ (except for $\theta = \pi/2$),

$$
\begin{aligned}
A &\simeq -\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta \\
B &\simeq -\frac{\omega_{pe}^2}{\omega^2}(1+\gamma)(1+\cos^2\theta) \\
C &\simeq -\frac{\omega_{pe}^2}{\omega^2}(1+\gamma)^2.
\end{aligned}
$$

Taking $(k_x, k_y, k_z) = (k_\perp, 0, k_\parallel)$ the wave equation can be rewritten as

$$\begin{pmatrix} 1+\gamma-n_\parallel^2 & 0 & n_\parallel n_\perp \\ 0 & 1+\gamma-n^2 & 0 \\ n_\parallel n_\perp & 0 & P-n_\perp^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

The solutions of this dispersion relation are

$$\begin{aligned} n^2 &= 1+\gamma \\ n_\parallel^2 &= 1+\gamma. \end{aligned}$$

Defining the Alfvén velocity $v_\mathrm{A} \equiv c/\sqrt{1+\gamma}$, these solutions can be rewritten as

$$\begin{aligned} \omega^2 &= k^2 v_\mathrm{A}^2 \\ \omega^2 &= k_\parallel^2 v_\mathrm{A}^2. \end{aligned}$$

Fast mode (compressional mode)

$$\begin{aligned} E_x &\simeq 0 \\ E_y &\neq 0 \\ E_z &\simeq 0 \\ \vec{v}_i &\sim \hat{x}\cos(k_x x + k_z z - \omega t). \end{aligned}$$

For this mode $\vec{k}\cdot\vec{v} \neq 0$, and is compressible $(\nabla\cdot\vec{v})$.

Slow mode (torsional mode or shear mode)

$$\begin{aligned} E_x &\neq 0 \\ E_y &\simeq 0 \\ E_z &\simeq 0 \\ \vec{v}_i &\sim \hat{y}\cos(k_x x + k_z z - \omega t). \end{aligned}$$

For this mode $\vec{k}\cdot\vec{v} = 0$, and is incompressible.

2.3 Ion Cyclotron Waves

The shear Alfvén wave becomes the ion cyclotron wave as ω approaches Ω_i. Still assuming $\omega \ll |\Omega_e|$ and $\gamma \gg 1$, $R \simeq \gamma\Omega_i/(\Omega_i+\omega)$ and $L \simeq \gamma\Omega_i/(\Omega_i-\omega)$. The dispersion relation is given by

$$n_\perp^2 = \frac{[\gamma\Omega_i - n_\parallel^2(\Omega_i+\omega)][\gamma\Omega_i - n_\parallel^2(\Omega_i-\omega)]}{\gamma\Omega_i^2 - n_\parallel^2(\Omega_i^2-\omega^2)}.$$

The shear Alfvén resonance occurs where $n_\perp^2 \to \infty$

$$\omega^2 = \Omega_i^2 \frac{c^2 k_\parallel^2}{c^2 k_\parallel^2 + \gamma\Omega_i^2} = \Omega_i^2 \frac{c^2 k_\parallel^2}{c^2 k_\parallel^2 + \omega_{pi}^2}.$$

An alternate form of dispersion relation is

$$n^4 \cos^2\theta - n^2 \frac{\gamma\Omega_i^2}{\Omega_i^2 - \omega^2}(1 + \cos^2\theta) + \frac{\gamma^2\Omega_i^2}{\Omega_i^2 - \omega^2} = 0.$$

For $\left|\Omega_i^2 - \omega^2\right| \ll \Omega_i^2$ ($i.e.$, $\omega \simeq \Omega_i$)

$$\begin{aligned} n^2 &= \frac{\gamma}{1 + \cos^2\theta} && \text{fast wave (FW)} \\ n^2\cos^2\theta &= \frac{\gamma\Omega_i^2}{\Omega_i^2 - \omega^2}(1 + \cos^2\theta) && \text{ion cyclotron wave (ICW).} \end{aligned}$$

For the ion cyclotron wave

$$\omega^2 = \Omega_i^2 \left(1 + \frac{\omega_{pi}^2}{c^2 k_\parallel^2} + \frac{\omega_{pi}^2}{c^2 k_\parallel^2 + c^2 k_\perp^2}\right)^{-1}.$$

In the presence of parallel gradients, $k_\parallel^2 \to \infty$ as $\omega \to \Omega_i$.

Polarization

$$\begin{aligned} &\text{ICW} \quad \frac{iE_x}{E_y} = -\frac{\Omega_i}{\omega\cos^2\theta} = -\frac{\Omega_i}{\omega}\left(1 + \frac{k_\perp^2}{k_\parallel^2}\right) \to -1 \quad (\omega \to \Omega_i) \\ &\text{FW} \quad \frac{iE_x}{E_y} \simeq \frac{(\omega/\Omega_i)^2 + \cos^2\theta}{(\omega/\Omega_i)(1 + \cos^2\theta)} \to 1 \quad (\omega \to \Omega_i) \end{aligned}$$

2.4 High Frequency Electromagnetic Waves

Solutions of the dispersion relation $An^4 - Bn^2 + C = 0$ can be written in the form

$$n^2 = 1 - \frac{2(A - B + C)}{2A - B \pm \sqrt{B^2 - 4AC}}.$$

In the high frequency limit $\omega \gg \Omega_i$, this can be expressed as

$$n^2 = 1 - \frac{2\omega_{pe}^2(\omega^2 - \omega_{pe}^2)/\omega^2}{2(\omega^2 - \omega_{pe}^2) - \Omega_e^2\sin^2\theta \pm \Omega_e\Delta}$$

where

$$\Delta = \left[\Omega_e^2 \sin^4\theta + 4\frac{(\omega^2 - \omega_{pe}^2)^2}{\omega^2}\cos^2\theta\right]^{1/2}.$$

This is called the Appleton-Hartree dispersion relation, and describes high frequency electromagnetic modes. Depending on the propagation angle, modes are classified as quasi-transverse (QT) and quasi-longitudinal (QL)

$$\begin{aligned} &\text{QT} \quad \Omega_e^2\sin^4\theta \gg 4\frac{(\omega^2 - \omega_{pe}^2)^2}{\omega^2}\cos^2\theta \\ &\text{QL} \quad \Omega_e^2\sin^4\theta \ll 4\frac{(\omega^2 - \omega_{pe}^2)^2}{\omega^2}\cos^2\theta \end{aligned}$$

Two modes exist for both QT and QL propagation

$$\text{QT-O} \qquad n^2 \simeq \frac{\omega^2 - \omega_{pe}^2}{\omega^2 - \omega_{pe}^2 \cos^2\theta}$$

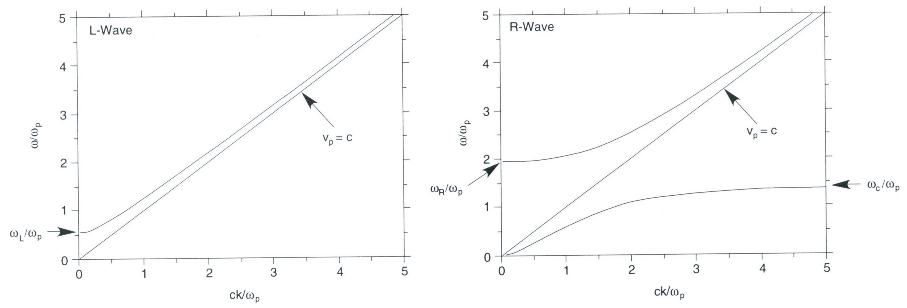
$$\text{QT-X} \qquad n^2 \simeq \frac{(\omega^2 - \omega_{pe}^2)^2 - \omega^2\Omega_e^2 \sin^2\theta}{\omega^2(\omega^2 - \omega_{pe}^2) - \omega^2\Omega_e^2 \sin^2\theta}$$

and for $\Omega_e^2 \sin^2\theta \ll \left|2(\omega^2 - \omega_{pe}^2)\right|$

$$\text{QL-R} \qquad n^2 \simeq 1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e \cos\theta)}$$

$$\text{QL-L} \qquad n^2 \simeq 1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_e \cos\theta)}.$$

The QL-R mode has a resonance at $\omega = |\Omega_e|$ for $\theta = 0$. This mode is called the electron cyclotron wave, and can propagate in the density range $\omega < \omega_{pe}$.



Dispersion relations of the L wave and the R wave.