11 Quasilinear Diffusion in a Magnetized Plasma

11.1 Cyclotron Heating

Consider the velocity change experienced by a single particle moving along $\vec{z}B_0(z)$ through a region of cyclotron resonance. Consider the case of ion cyclotron resonance. The electric field component that accelerates ions is the left-hand circularly polarized component (E^+). The resonance condition is

$$\omega - k_\parallel v_\parallel - n\Omega = 0$$

which means that the Doppler-shifted frequency seen by the particle, travelling along B_0 at $v_\parallel$, is equal to an integral harmonic of the ion cycloton frequency. The time variation in $B_0(t)$, the magnetic field at the particle location, arises from the motion of the particle along B_0. Defining $u = v_x + iv_y$, expanding

$$\Omega(t) = \omega + (t - t_{\rm res})\Omega'$$

and ignoring the nonresonant driving term by E^-, the single particle equation of motion can be written as

$$\frac{du}{dt} + i\Omega(t)u = \frac{q}{m}E^+e^{-i\omega t}$$

which can be integrated to

$$u(t)\exp\left[i\int_{t_0}^{t} dt'\,\Omega(t')\right] = u(t_0) + \frac{q}{m}e^{-i\psi}E^+\left|\frac{2\pi}{\Omega'}\right|^{1/2}$$

where

$$\psi = \omega t_0 + \frac{\Omega'}{2}(t_{\rm res} - t_0)^2 - \frac{\pi}{4}{\rm sgn}(\Omega').$$

The average change in energy per transit is

$$W_\perp = \frac{m_s}{2}\left\langle u(t)u(t)^* - u(t_0)u(t_0)^*\right\rangle = \frac{m_s}{2}\left|\frac{Z_seE^+}{m_s}\right|^2\left|\frac{2\pi}{\Omega'}\right|.$$

The rate of power absorption per unit volume is $\nu(\vec{r}, v_\perp, v_\parallel)W_\perp(\vec{r}, v_\perp, v_\parallel)$ where $\nu(\vec{r}, v_\perp, v_\parallel)$ is the number of transits of particles across resonant surfaces per unit volume and per unit time.

$$P_\perp = \frac{\pi Z_s^2 e^2}{m_s|k_\parallel|}\left|E^+\right|^2 n_{\rm res}(\vec{r}, v_\parallel^{\rm res})$$

where $n_{\rm res}$ is the density of resonant particles.

11.2 Heating in Tokamak Geometry

In a tokamak plasma the magnetic field varies approximately as $1/R$ where R is the distance from the axis of symmetry (*i.e.*, major radius). The location of resonance depends on $v_\parallel$ of each particle, in order to satisfy $\omega - k_\parallel v_\parallel = \Omega(R)$. The number of resonant particles can be written

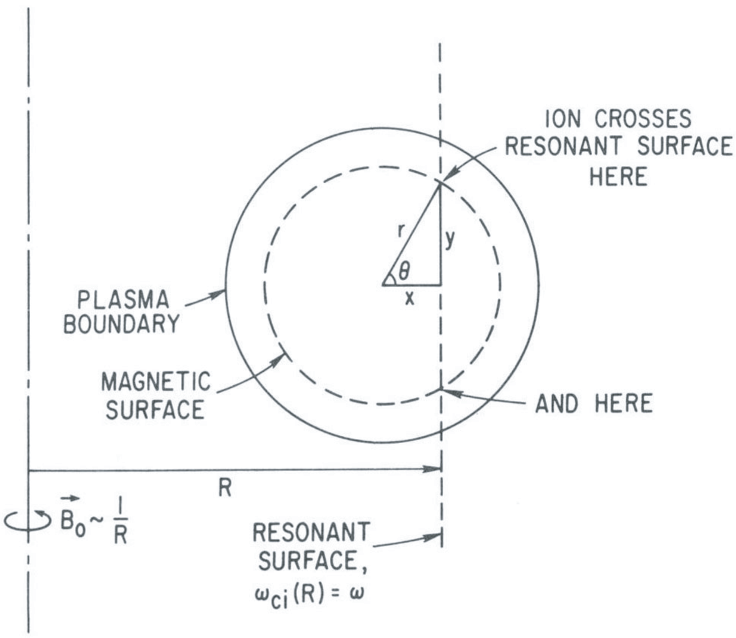
$$n_{\rm res} = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty 2\pi v_\perp dv_\perp f(\vec{r}, v_\perp, v_\parallel)\delta\left(v_\parallel - \frac{\omega - \Omega}{k_\parallel}\right)$$

so that

$$\int_{R_1}^{R_2} dR\, n_{\rm res}(\vec{r}) = n(r)\left|\frac{k_\parallel}{d\Omega/dR}\right| .$$

The spread of the resonance region is $\Delta R \simeq k_\parallel v_{th}/(d\Omega/dR) \simeq k_\parallel \rho_L R$. Averaging $P_\perp$ over the region between two magnetic surfaces, the average power per unit volume to particles is

$$\langle P_\perp \rangle = \frac{n_s(r) Z_s e}{B_0(R_0)} \frac{R_0[1 + (r/R_0)\cos\theta]^3}{r|\sin\theta|} \left|E^+\right|^2 .$$



Fg. 1. Geometry for ion cyclotron resonance in a tokamak.

11.3 Quasilinear Diffusion in a Magnetic Field

For the case of ion cyclotron acceleration by an electric field with pure left-hand circular polarization,

$$\begin{aligned} (\Delta v_\perp)^2 &= |v^+|^2 = 4|A^+|^2 \frac{\sin^2(\beta t/2)}{\beta^2} \\ &\simeq 2\pi |A^+|^2 t\delta(\beta) \end{aligned}$$

where

$$\beta = \omega - k_\parallel v_\parallel - \Omega$$

$$A^+ = \frac{Ze}{m}\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right)E^+.$$

The diffusion coefficient is

$$D_\perp = \frac{\overline{(\Delta v_\perp)^2}}{2t} = \pi|A^+|^2\delta(\omega - k_\parallel v_\parallel - \Omega).$$

11.4 Electromagnetic Quasilinear Theory

The relativistic Vlasov equation is

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f + \frac{q}{m}\nabla_{\vec{p}}\cdot\left[\left(\vec{E} + \vec{v}\times\vec{B}\right)f\right] = 0.$$

The equation for quasilinear evolution of f derived by Kennel and Engelmann is

$$\frac{\partial f_0(p_\perp, p_\parallel, t)}{\partial t} = \lim_{V\to\infty}\pi q^2\sum_{n=-\infty}^{\infty}\int\frac{d^3\vec{k}}{V}Lp_\perp\delta(\omega - k_\parallel v_\parallel - n\Omega)|\psi_{n,k}|^2 p_\perp L f_0$$

where

$$L = \left(1 - \frac{k_\parallel v_\parallel}{\omega_{kr}}\right)\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp} + \frac{k_\parallel v_\parallel}{\omega_{kr}}\frac{1}{p_\perp}\frac{\partial}{\partial p_\parallel}$$

$$\psi_{n,k} = \frac{1}{2}\left(E_{kx} + iE_{ky}\right)e^{-i\theta}J_{n-1}(z) + \frac{1}{2}\left(E_{kx} - iE_{ky}\right)e^{i\theta}J_{n+1}(z) + \frac{p_\parallel}{p_\perp}E_{kz}J_n(z)$$

$z = k_\perp v_\perp/\Omega$ and $\vec{k} = (k_\perp\cos\theta, k_\perp\sin\theta, k_\parallel)$.

11.5 Cyclotron Frequency Heating

In the nonrelativistic limit, the power absorbed is

$$P_\perp = 2\pi\int v_\perp dv_\perp dv_\parallel\left(\frac{m_s v_\perp^2}{2}\right)\frac{\partial f_0(v_\perp, v_\parallel)}{\partial t} = \frac{\pi Z_s^2 e^2}{4m_s|k_\parallel|}\left|E_x \pm i\mathrm{sgn}(\omega)E_y\right|^2 n_{\mathrm{res}}$$

where the density of resonant particles is

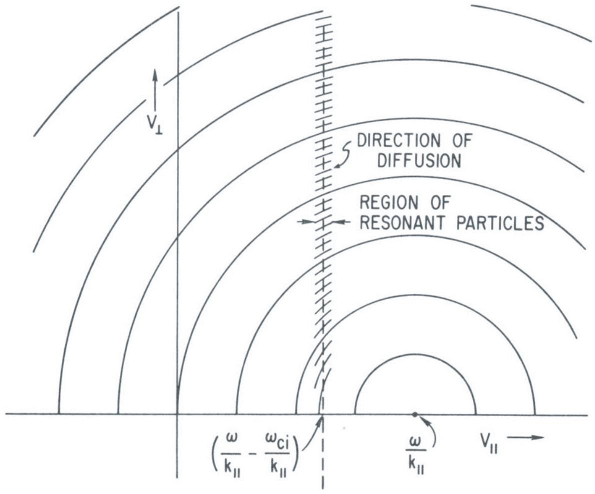
$$n_{\mathrm{res}} = 2\pi\int v_\perp dv_\perp dv_\parallel\delta(v_\parallel - \frac{\omega - |\Omega|}{k_\parallel})f_0(v_\perp, v_\parallel, t).$$

11.6 Resonant Particle Diffusion

In the nonrelativistic limit, $LF(v_\perp, v_\parallel) = 0$ where $F = F[v_\perp^2 + (v_\parallel - \omega_{kr}/k_\parallel)^2]$. Therefore, the L operator is a gradient operator $\nabla_v f_0(v_\perp, v_\parallel)$ along the circles

$$v_\perp^2 + \left(v_\parallel - \frac{\omega_{kr}}{k_\parallel}\right)^2 = \mathrm{const.}$$

Quasilinear diffusion occurs along these circles.



Fg. 2. Contours for quasilinear diffusion. Diffusion occurs for only resonant particles and only in the circumferential direction.

11.7 Steady State Solution for $f(\vec{v})$

The isotropic steady state solution $f(v)$ of the Fokker-Planck equation

$$\frac{\partial f}{\partial t} = C(f) + Q(f)$$

is given by

$$\ln f(v) = -\frac{E}{T_e(1+\xi)}\left[1 + \frac{R_f(T_e - T_f + \xi T_e)}{T_f(1 + R_f + \xi)} K\left(\frac{E}{E_f}\right)\right]$$

where $E = mv^2/2$, $R_f = n_f Z_f^2 l_f / n_e l_e$, $l_f = \sqrt{m_f/2T_f}$, $\epsilon = 2/3\sqrt{\pi}$,

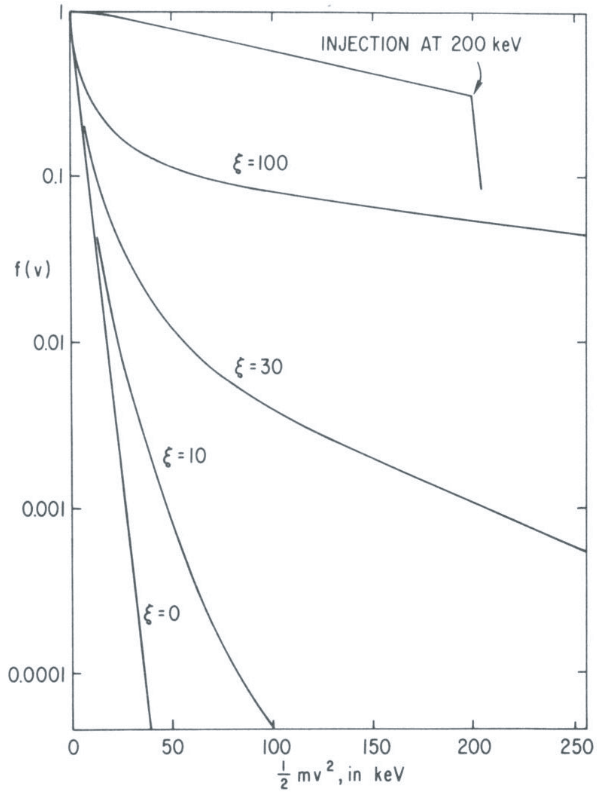
$$\xi = \frac{1}{3}\frac{\langle P_\perp \rangle t_s}{nT_e},$$

$$E_f(\xi) = \frac{mT_f}{m_f}\left[\frac{1 + R_f + \xi}{2\epsilon(1+\xi)^{2/3}}\right],$$

$$K(x) = \frac{1}{x}\int_0^x \frac{du}{1 + u^{3/2}},$$

$\langle P_\perp \rangle$ is the wave heating power per unit volume averaged over a flux surface, and t_s is the slowing down time. It is useful to define an effective temperature $T_{eff} = -(d \ln f/dE)^{-1}$ at each value of $E = mv^2/2$

$$\frac{1}{T_{eff}} = \frac{1}{T_e(1+\xi)}\left[1 + \frac{R_f(T_e - T_f + \xi T_e)}{T_f(1 + R_f + \xi)} \frac{1}{1 + (E/E_f)^{3/2}}\right].$$



Fg. 3. Plots of $f(v)$ for different levels of ξ. Contours for quasilinear diffusion. Diffusion occurs for only resonant particles and only in the circumferential direction.