1 Description of Waves

1.1 Cold Plasma Model

Ignore thermal motion, *i.e.*, assume that $T_\mathrm{e} = T_\mathrm{i} = 0$. For simplicity, further assume:

- $n_\mathrm{e} = Zn_\mathrm{i}$
- frictionless, homogeneous plasma
- uniform static $\vec{B}$

1.2 Susceptibility and Dielectric Tensor

$$\nabla \times \frac{\vec{B}}{\mu_0} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{j} = \frac{\partial \vec{D}}{\partial t}$$

Electric displacement $\vec{D}$ is defined as

$$\begin{aligned} \vec{D}(\omega, \vec{k}) &= \epsilon_0 \overleftrightarrow{\epsilon}(\omega, \vec{k}) \cdot \vec{E}(\omega, \vec{k}) \\ &= \epsilon_0 \left[\vec{E}(\omega, \vec{k}) + \frac{i}{\omega \epsilon_0} \vec{j}(\omega, \vec{k}) \right] \end{aligned}$$

where $\overleftrightarrow{\epsilon}$ is the dielectric tensor (dimensionless)

$$\overleftrightarrow{\epsilon}(\omega, \vec{k}) = \overleftrightarrow{1} + \sum_s \overleftrightarrow{\chi}_s(\omega, \vec{k})$$

where $s = \mathrm{e}, \mathrm{i}$ is the species index, and $\overleftrightarrow{\chi}_s$ is the susceptibility tensor. The induced current $\vec{j}$ can be related to the wave electric field $\vec{E}$ by

$$\vec{j} = \sum_s \vec{j}_s = \sum_s n_s q_s \vec{v}_s$$

$$\vec{j}_s = \overleftrightarrow{\sigma}_s \cdot \vec{E} = -i\omega\epsilon_0 \overleftrightarrow{\chi}_s \cdot \vec{E}$$

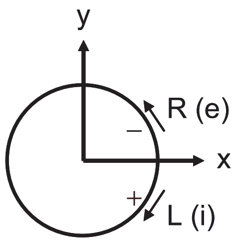
where $\overleftrightarrow{\sigma}_s$ is the conductivity tensor.

Equation of motion for species s is written as

$$n_s m_s \frac{d\vec{v}_s}{dt} = n_s m_s \left(\frac{\partial \vec{v}_s}{\partial t} + \vec{v}_s \cdot \nabla \vec{v}_s \right) = n_s q_s \left(\vec{E} + \vec{v}_s \times \vec{B} \right) - \nabla \cdot \overleftrightarrow{\Psi}_s .$$

The left hand side is called the convective derivative, and $\overleftrightarrow{\Psi}_s$ is the stress tensor. In the cold plasma approximation, $\overleftrightarrow{\Psi}_s = 0$. Taking the static magnetic field to be in the z-direction, $\vec{B}_0 = B_0 \hat{z}$, and using the left hand and right hand rotating coordinates such as $\vec{v}^\pm = (v_x \pm i v_y)/2$,

$$\begin{aligned} v_s^\pm &= \frac{iq_s}{m_s} \frac{E^\pm}{\omega \mp \Omega_s} \\ v_{zs} &= \frac{iq_s}{m_s} \frac{E_z}{\omega} \end{aligned}$$



Rotating coordinates $(x \pm iy)/2$.

where $\Omega_s \equiv q_s B_0/m_s$ is the cyclotron frequency (note that $\Omega_e < 0$).

The wave induced current can be written

$$j_s^\pm = n_s q_s v_s^\pm = -i\omega\epsilon_0\chi_s^\pm E^\pm$$

$$\chi_s^\pm = -\frac{\omega_{ps}^2}{\omega(\omega \mp \Omega_s)}.$$

Other elements of $\overleftrightarrow{\chi}_s$ are given by

$$\chi_{xx,s} = \chi_{yy,s} = \frac{\chi_s^+ + \chi_s^-}{2}$$

$$\chi_{xy,s} = -\chi_{yx,s} = \frac{i(\chi_s^+ - \chi_s^-)}{2}$$

$$\chi_{zz,s} = -\frac{\omega_{ps}^2}{\omega^2}$$

where ω_{ps} is the plasma frequency defined as $\omega_{ps}^2 \equiv n_s q_s^2/(\epsilon_0 m_s)$.

The cold plasma dielectric tensor $\overleftrightarrow{\epsilon}$ can be expressed as

$$\overleftrightarrow{\epsilon} \cdot \vec{E} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

using the following Stix parameters

$$S = \frac{R+L}{2}$$
$$D = \frac{R-L}{2}$$

$$R = 1 + \sum_s \chi_s^- = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \Omega_s)}$$
$$L = 1 + \sum_s \chi_s^+ = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \Omega_s)}$$
$$P = 1 + \sum_s \chi_{zz,x} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

1.3 Dispersion Relation

Maxwell equations

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \frac{\vec{B}}{\mu_0} &= \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{j} = \epsilon_0 \frac{\partial}{\partial t}\left(\overleftrightarrow{\epsilon} \cdot \vec{E}\right)\end{aligned}$$

can be combined to give the homogeneous plasma wave equation

$$\vec{k} \times \left(\vec{k} \times \vec{E}\right) + \frac{\omega^2}{c^2} \overleftrightarrow{\epsilon} \cdot \vec{E} = 0.$$

Introducing the index of refraction $\vec{n} \equiv c\vec{k}/\omega$, this can be written as

$$\vec{n} \times \left(\vec{n} \times \vec{E}\right) + \overleftrightarrow{\epsilon} \cdot \vec{E} = 0.$$

In the coordinate system with $\vec{B}_0 = (0,\, 0,\, B_0)$ and $\vec{n} = (n \sin\theta,\, 0,\, n\cos\theta)$,

$$\begin{pmatrix} S - n^2\cos^2\theta & -iD & n^2\cos\theta\sin\theta \\ iD & S - n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P - n^2\sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0.$$

The condition that this equation has a solution for finite $\vec{E}$, $\det(\;) = 0$, yields the dispersion relation, which is a relationship between ω and $\vec{k}$ that the wave must satisfy. The cold plasma dispersion relation can be written as

$$An^4 - Bn^2 + C = 0$$

where

$$\begin{aligned}A &= S\sin^2\theta + P\cos^2\theta \\ B &= RL\sin^2\theta + PS(1+\cos^2\theta) \\ C &= PRL.\end{aligned}$$

There are two solutions to the dispersion relation

$$n^2 = \frac{B \pm \sqrt{(RL-PS)^2\sin^4\theta + 4P^2D^2\cos^2\theta}}{2A}$$

or alternatively

$$\tan^2\theta = \frac{-P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}.$$

For $\theta = 0$ (propagation parallel to $\vec{B}_0$), the solutions are $P = 0$, $n^2 = R$ and $n^2 = L$. For $\theta = \pi/2$ (propagation perpendicular to $\vec{B}_0$), the solutions are $n^2 = RL/S$ and $n^2 = P$.

1.4 Polarization and Phase Relations

It is convenient to use complex notation for linear waves.

$$\vec{E}(\vec{r}, t) = \Re\left[\vec{E}(\vec{k}, \omega)\exp(i\vec{k}\cdot\vec{r} - i\omega t)\right]$$

where $\Re[\ ]$ indicates real part of a complex expression. We adopt the convention that $\omega > 0$. Consider the electric field at a fixed position, $\vec{r} = 0$. The x-component of the electric field can be expressed as $E_x = a\exp(-i\omega t)$ so that $\Re(E_x) = a\cos(-\omega t) = a\cos(\omega t)$. If $E_y = iE_x$, $\Re(E_y) = -a\sin(-\omega t) = a\sin(\omega t)$. The motion is in the right-hand $(+\theta)$ direction when $iE_x/E_y = +1$. Similarly, when $iE_x/E_y = -1$, the motion is in the left-hand $(-\theta)$ direction, so iE_x/E_y can be used to describe polarization.

The y-component of the wave equation can be written

$$iDE_x + (S - n^2)E_y = 0$$

from which one obtains

$$\frac{iE_x}{E_y} = \frac{n^2 - S}{D}.$$

The two modes of propagation parallel to the field $(\theta = 0)$ are polarized as follows

$$n^2 = R \qquad \frac{iE_x}{E_y} = 1 \qquad \text{right-hand (e)}$$

$$n^2 = L \qquad \frac{iE_x}{E_y} = -1 \qquad \text{left-hand (i).}$$

Similarly polarization for particle velocity can be defined

$$\frac{iv_{x,s}}{v_{y,s}} = -\frac{(\omega + \Omega_s)(n^2 - R) + (\omega - \Omega_s)(n^2 - L)}{(\omega + \Omega_s)(n^2 - R) - (\omega - \Omega_s)(n^2 - L)}$$

$$n^2 = R \qquad \frac{iv_{x,s}}{v_{y,s}} = 1 \qquad \text{right-hand}$$

$$n^2 = L \qquad \frac{iv_{x,s}}{v_{y,s}} = -1 \qquad \text{left-hand}$$

1.5 Cutoff and Resonance

Cutoff $(n^2 \to 0)$

From the dispersion relation $An^4 - Bn^2 + C = 0$, $C = 0$ must be satisfied at cutoff, *i.e.*, $P = 0$, $R = 0$ or $L = 0$.

Propagation and Evanescence

The solution of the dispersion relation n^2 is real when $B^2 - 4AC > 0$. In this case n is either purely real or purely imaginary. When $n^2 > 0$, n is purely real and the solution describes a propagating wave, $\exp(ik_{\rm re}x - i\omega t)$. When $n^2 < 0$,

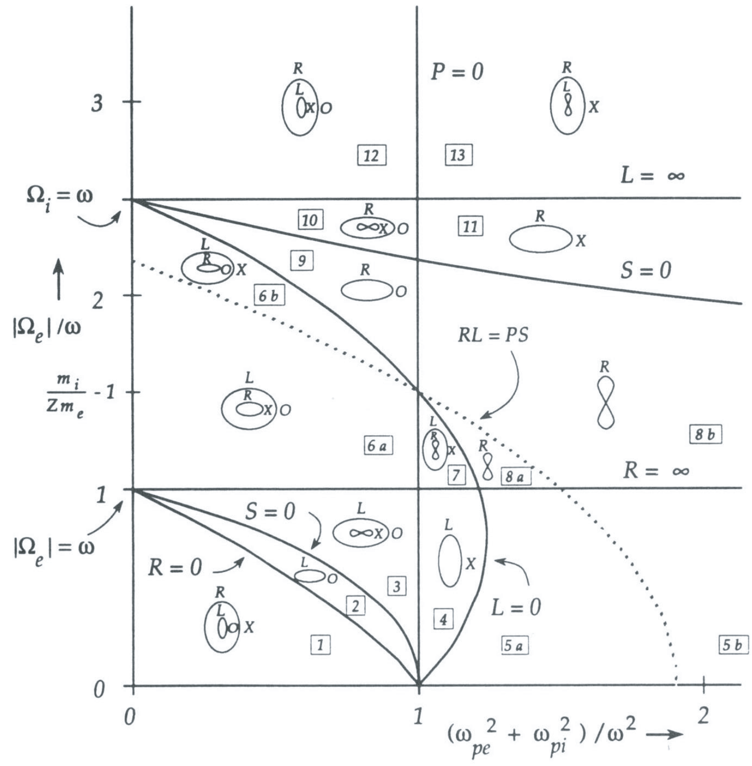
n is purely imaginary and the solution describes a non-propagating evanescent mode which decays exponentially, $\exp(-k_{\rm im}x - i\omega t)$.

Resonance ($n^2 \to \infty$)

From the dispersion relation $A = 0$ must be satisfied at resonance, *i.e.*, $\tan^2\theta = -P/S$. This means resonance depends on the angle of propagation. Resonances corresponding to $\theta = 0$ and $\theta = \pi/2$ are called principal resonances. At $\theta = 0$, $S \to \infty$, which can happen when $R \to \infty$ (electron cyclotron resonance) or $L \to \infty$ (ion cyclotron resonance). At $\theta = \pi/2$, resonances occur at $S = 0$ (hybrid resonances).

1.6 CMA Diagram

The CMA (Clemmow-Mullaly-Allis) diagram describes all cold plasma waves. The normalized phase velocity vector $\vec{v}_{\rm ph}/c = \hat{k}\omega/(ck)$ is displayed schematically in a polar plot in each region of propagation separated by bounding surfaces (cutoffs and resonances). Its magnitude $u = 1/n$ satisfies the dispersion relation $Cu^4 - Bu^2 + A = 0$.



CMA diagram.